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On the extrication of large objects from the ocean bottom (the breakout phenomenon)

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An analytical theory is developed to describe how negative pressure, (or 'mud suction', as it is sometimes referred to) develops underneath a body as it detaches itself from the ocean bottom. Biot's quasistatic equations of poro-elasticity are used to model the ocean bottom, and a general three-dimensional time-dependent analysis of the problem is worked out first using the boundary-layer approximation recently proposed by Mei and Foda. Then, explicit leading-order analytical solutions are presented for the problems of extrication of slender bodies as well as axisymmetric bodies from the ocean bottom.

1. Introduction

The extrication of objects (e.g. sunken vessels, construction caissons) from the ocean bottom is a problem whose importance in ocean and offshore engineering has only recently been realized, as a result of the growing engineering activities in the offshore region. It is the task of salvage engineers to develop efficient and successful techniques for carrying out such extrication. It is well known (Liu 1969) that actual salvage operations may last several days in trials, and that sometimes the object never even breaks out at all owing to the engineers' lack of understanding of the 'breakout' phenomenon. The phenomenon is also important for the design and operation of submersibles in the ocean. For example, a submersible design engineer would be interested in knowing the minimum propulsion power required for the submersible to break loose from the bottom mud after resting on it for a period of time. Indeed, as in the problem of detachment of two plates initially in contact at one of their plane surfaces, the required release force might be very large (infinite in the limiting case of two rigid impervious plates with no initial gap between them). The release force in excess of the submerged weight of the object is usually called the 'breakout force' and sometimes referred to as the 'mud suction'. From salvage experience, it is known that, beside the magnitude of the breakout force, the length of time this force has to be applied is of equal importance, i.e. the phenomenon is time-dependent. In order to achieve complete detachment of the object from the soil, the pulling force has to be applied for a certain period of time called the 'breakout time'. Reliable estimates of the breakout force and time are only possible after one understands the mechanism of the breakout phenomenon, and consequently defines a criterion for the breakout time.

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Obtaining such estimates would eliminate the loss of time and equipment during actual lifting operations. Moreover, reliable estimates become essential in submarine rescue missions where the breakout time is limited by the submarine emergency life-support capacity, and a corresponding breakout force should be estimated before the rescue mission is launched to ensure that the force is within the capacity of the surface ship.

Very little knowledge concerning the breakout phenomenon can be found in the literature and only traces of salvage experience are recorded occasionally in ships' logs. Unpublished preliminary laboratory studies of the breakout phenomenon were conducted by Harleman (1963, personal communication). A few other experimental studies in the field and in the laboratory were also conducted in the late nineteen sixties (see e.g. Liu 1969; Muga 1968; DeHart & Ursell 1967) in order to obtain some empirical knowledge of the breakout force–time relation. However, the data obtained were so limited and so scattered that none of these studies permitted the making of any affirmative conclusion from them. Only some guidelines can be drawn based on these investigations. An important one is the observation in all experiments that the characteristics of the porous soil (e.g. soil permeability and soil firmness) affect the breakout process significantly. Therefore, analysis of the problem should account adequately for the response of the saturated porous ground.

Biot (1941) presented a three-dimensional consolidation theory for fluid-filled porous media which takes into account the deformation of the porous solid and the flow of the pore fluid in a physically consistent way. In the present paper, Biot's theory will be employed to model the ocean bottom. The basic assumptions involved are that the soil solid skeleton is linearly elastic and that the pore fluid obeys Darcy's law. However, since the governing equations of Biot couple the pore pressure with the solid displacements, the mathematical treatment of these equations is very difficult, and indeed there are very few exact solutions to Biot's equations in the literature (see Yamamoto *et al.* 1978; Verruijt 1969). Therefore, a boundary-layer approximation recently worked out by Mei & Foda (1980, 1981) will be used to obtain an approximate 'small-time' solution to Biot's equations. It will be shown that, for a wide range of physical parameters, the breakout phenomenon is well within the range of the small-time boundary-layer formulation. A general time-dependent three-dimensional analysis of the problem will be worked out first, then explicit analytical solutions will be presented for the problems of extrication of slender bodies as well as axisymmetric bodies from the ocean bottom.

2. Formulation of the problem

Cartesian co-ordinates (x_1, x_2, x_3) will be used, with the x_3 axis taken positive upward. Let the porous bed occupy the lower half-space $x_3 \leqslant 0$ and let a three-dimensional rigid body with a plane base be seated on the horizontal ocean floor $x_3 = 0$, with the origin O of the co-ordinates conveniently located at the centroid of the base. Initially, the body is neutrally buoyant while its base is still touching the ocean floor, meaning that the pulling force acting on the body is initially equal to the body's submerged weight. The entire ocean floor, including the area underneath the base, is therefore stress-free. The problem of interest here is to investigate what will happen when the body is uniformly pulled up away from the ocean floor at a velocity $W(t)$ without tilt.

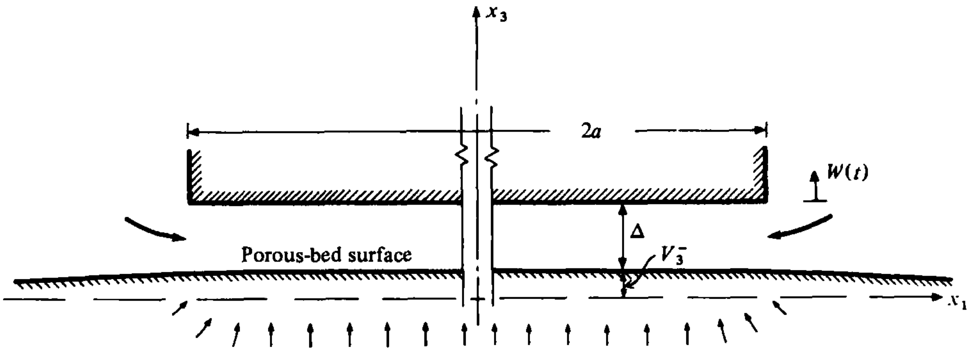

FIGURE 1. Definition sketch. As the gap between the body and the porous bed expands, water flows into the gap laterally through the gap periphery as well as vertically from the bed's pores. The porous-bed upward deflection is due to the viscous drag exerted by the pore fluid flowing upward into the gap.

As the body begins to move upward a tiny gap filled with water will begin to develop between the body and the porous free surface (figure 1). The governing equations will be statements of conservation of mass and momentum in the fluid gap as well as in the saturated ground.

First, we examine the importance of inertia in the gap and ground motions. In the gap, inertia is negligible when the Reynolds' number is very small, i.e. when

$$R = \frac{\rho U_1 \Delta}{\mu} \ll 1, \tag{1}$$

where U_1 is the typical horizontal water velocity in the gap, Δ is the gap thickness, and μ and ρ are respectively the water viscosity and density. As for the ground response, inertia may also be neglected if the body velocity W is applied smoothly, so that at no time is W comparable to the phase velocity of the saturated-soil elastic waves. This condition can easily be met in reality, as will be seen later in the analysis. Therefore, it will be assumed that condition (1) holds and that inertia effects in the ground are negligible as well.

Since the gap thickness Δ is much smaller than the horizontal dimension of the gap (say a), a lubrication-type approximation (Lamb 1932, p. 581), is employed to describe the motion there:

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (i = 1, 2, 3) \quad \text{(continuity)}, \tag{2a}$$

$$\frac{\partial p}{\partial x_i} = \mu \frac{\partial^2 u_i}{\partial x_3^2} \quad (i = 1, 2), \quad \frac{\partial p}{\partial x_3} = 0 \quad \text{(momentum)}, \tag{2b}$$

where p is the excess pressure over its hydrostatic value and u_i is the fluid velocity. The appropriate boundary conditions for (2) are as follows.

(i) At the periphery of the gap

$$p = 0. \tag{3}$$

Strictly speaking, this boundary condition should be imposed at $|x_1|, |x_2| = \infty$ and the effects of fluid flow outside the gap should be considered. However, such inertia-free external flow, having the small gap thickness Δ as its characteristic length scale, is expected to have a negligible effect on the pressure and is ignored here.

(ii) At the body–gap interface

$$u_1 = u_2 = 0, \quad u_3 = W(t) \quad \text{at} \quad x_3 = \int_0^t W dt. \qquad (4a, b, c)$$

(iii) At the gap–porous-bed interface we shall impose the conditions that the three velocity components and the three stress components are continuous across the interface. The velocity-continuity condition requires that

$$u_i^+ = nu_i^- + (1-n)v_i^- \quad \text{at} \quad x_3 = \int_0^t v_3^- \, dt, \qquad (5a, b, c)$$

where n is the porosity (i.e. the water per unit cross-sectional area in the saturated porous bed) and v_i is the solid velocity in the porous bed. The superscript + means just above the interface, while − means just below it.

Stress continuity requires (only excess dynamical components are considered)

$$\tau_{ij}^- N_j = \tau_{ij}^+ N_j, \qquad (6)$$

where N_j is the unit normal to the bed surface, and τ_{ij}^- and τ_{ij}^+ are the second-order stress tensors below and above the interface. Above the interface, the stress tensor in the water is given by

$$\tau_{ij}^+ = -p^+\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^+. \qquad (7)$$

However, the ratios $(N_1, N_2)/N_3$ are $O(\Delta/a)$ or smaller. Therefore, in order to be consistent with the lubrication-theory assumption ($\Delta/a \ll 1$), we approximate the unit normal vector by the unit vertical vector, so that (6) becomes

$$\tau_{i3}^- = -p^+\delta_{i3} + \mu \left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right)^+. \qquad (8a, b, c)$$

Beside conditions (8), we further require that the water pressure is continuous across the interface, i.e.

$$p^- = p^+. \qquad (9)$$

It is these last seven boundary conditions, ($5a, b, c$), ($8a, b, c$) and (9), that couple the motion in the gap with the complicated fluid–solid flow system in the ground.

However, a compensating simplification of the problem arises from the fact that most soil materials have low permeabilities (see e.g. Terzaghi & Peck (1948, p. 48) for typical soil permeabilities, and Jaeger & Cook (1974) for rock permeabilities). Mei & Foda (1981) presented a boundary-layer formulation, based on Biot's (1941, 1956) three-dimensional consolidation theory, which takes advantage of the low-permeability assumption. They reasoned that, since soil of low permeability resists water flow relative to the soil solid matrix, it follows that motion characterized by sufficiently small time scale should have the following boundary-layer structure. Relative fluid–solid motion is appreciable in thin boundary layers near free surfaces, where it is much easier for the fluid to squeeze into or out of the free surface. Outside the boundary layers, such relative motion is highly resisted and both the fluid and the solid move together with essentially the same velocities. An approximate, leading-order solution is therefore written as the sum of the outer solution plus a boundary-layer correction. In the outer solution, no relative fluid–solid motion is allowed. A boundary-layer solution is then added to the outer solution in the region near the free surface in order to correct for possible relative motion there. The complete derivation of this boundary-layer formulation is given in Mei & Foda (1981) and will not be repeated here. For convenience, a brief summary of the procedure is given in the appendix. The formulations of Mei & Foda will be used here to obtain a small-time solution to the saturated ground response. Thus, we write the solution in the ground as the sum of the outer solution and the boundary-layer correction, i.e.

$$(\quad) = (\quad)^{\mathrm{o}} + (\quad)^{\mathrm{b}}. \tag{10}$$

The outer problem, $(\quad)^{\mathrm{o}}$, is stated as follows:

$$u_i^{\mathrm{o}} = v_i^{\mathrm{o}}, \quad \frac{\partial \tau_{ij}^{\mathrm{o}}}{\partial x_j} = 0, \tag{11), (12}$$

$$\frac{\partial p^{\mathrm{o}}}{\partial t} = -\frac{\beta}{n}\frac{\partial v_i^{\mathrm{o}}}{\partial x_i}, \tag{13}$$

$$\frac{\partial \tau_{ij}^{\mathrm{o}}}{\partial t} = G\left(\frac{\partial v_i^{\mathrm{o}}}{\partial x_j} + \frac{\partial v_j^{\mathrm{o}}}{\partial x_i} + \frac{2\nu_{\mathrm{e}}}{1-2\nu_{\mathrm{e}}}\delta_{ij}\frac{\partial v_k^{\mathrm{o}}}{\partial x_k}\right), \tag{14}$$

where β is the bulk modulus of the pore fluid, G is the shear modulus of the solid matrix, and ν_{e} is the equivalent Poisson's ratio of the composite fluid–solid system. The relation between ν_{e} and the actual Poisson's ratio ν of the solid matrix alone is given by

$$\nu_{\mathrm{e}} = \frac{1}{2}\left(\frac{2\nu}{1-2\nu} + \frac{\beta}{nG}\right)\left(\frac{1}{1-2\nu} + \frac{\beta}{nG}\right)^{-1}. \tag{15}$$

The reason for considering the pore-fluid compressibility β^{-1}, although it has been neglected for the water flow in the gap above (equation (2*a*)), is to account for possible entrapment of tiny pockets of air in the pores, which would greatly increase the compressibility of the water–air mixture. Therefore β should be considered as the apparent bulk modulus of the pore fluid, and is related to the bulk modulus β_0 of pure water by (Verruijt 1969)

$$\frac{1}{\beta} = \frac{1}{\beta_0} + \frac{1-S}{p_0}, \tag{16}$$

where S denotes the degree of saturation and p_0 denotes the absolute pore pressure ($\beta_0 = 10^9\,\mathrm{N/m^2}$ while, for 99 % saturation and $p_0 = 1\,\mathrm{atm}$, β reduces to $10^6\,\mathrm{N/m^2}$). The case of complete saturation will be considered here as a special case.

Equation (11) simply states that the outer fluid and solid velocities are equal. Equation (12) is the conventional equilibrium equation of Cauchy. Since the fluid motion follows the solid motion in the outer problem, the outer pore pressure p^{o} is therefore related to the solid dilatation as given in (13). Equation (14) is the conventional Hooke's law, with G and ν_{e} replacing G and ν, and is relating the total stress τ_{ij}^{o} with solid strains. By contracting the tensor equation (14), we get a relation between the solid dilatation and the trace of the stress tensor τ_{kk}^{o}, so that (13) may be rewritten as

$$p^{\mathrm{o}} = -\frac{(1-2\nu_{\mathrm{e}})\beta}{2(1+\nu_{\mathrm{e}})\,nG}\tau_{kk}^{\mathrm{o}}. \tag{17}$$

On the other hand, the boundary-layer correction $(\ )^{\mathrm{b}}$ is given by

$$\frac{\partial p^{\mathrm{b}}}{\partial x_i} = -\frac{n}{k}(u_i^{\mathrm{b}} - v_i^{\mathrm{b}}) = -\frac{n}{k}(u_i - v_i), \tag{18}$$

$$K\frac{\partial^2 p^{\mathrm{b}}}{\partial x_3^2} = \frac{\partial p^{\mathrm{b}}}{\partial t}, \quad K = k\left[\frac{n}{\beta} + \frac{1-2\nu}{2G(1-\nu)}\right]^{-1}, \tag{19}$$

$$\tau_{11}^{\mathrm{b}} = \tau_{22}^{\mathrm{b}} = -\frac{1-2\nu}{1-\nu}p^{\mathrm{b}}(1+O(\epsilon)), \tag{20}$$

$$\tau_{33}^{\mathrm{b}} = O(\epsilon), \quad \tau_{ij}^{\mathrm{b}} = O(\epsilon) \quad (i \neq j), \tag{21}$$

$$(v_i^{\mathrm{b}}, u_i^{\mathrm{b}}) = O(\epsilon)(v_i^{\mathrm{o}}, u_i^{\mathrm{o}}), \tag{22}$$

$$\epsilon = \frac{\delta}{a} = \frac{(Kt)^{\frac{1}{2}}}{a} \ll 1, \tag{23}$$

where k is the soil coefficient of permeability and a is the horizontal dimension of the gap. Equation (18) is Darcy's law, which states that frictional stresses in the porous bed are proportional to the relative fluid–solid velocities. Equation (19) governs the boundary-layer correction p^{b} to the pore pressure. Notice that (19) is a one-dimensional heat-diffusion equation and clearly from this equation the boundary-layer thickness is of order $\delta = (Kt)^{\frac{1}{2}}$. Therefore this boundary-layer formulation is only valid when $\delta \ll a$, as stated in (23). Notice that δ increases with time as $t^{\frac{1}{2}}$, and from (19) one sees that the smaller the permeability coefficient k the longer the time range of validity of (23). For example, if the ocean sediment is a composition of fine sands, silt or clay, with $k \simeq 10^{-9}$–$10^{-11}\,\mathrm{m^3\,s/kg}$, then the time range of validity according to (23) can be as long as a few hours, but it will be greatly reduced to just a few seconds if the ocean sediment is composed instead of highly permeable coarse sand with $k \simeq 10^{-6}\,\mathrm{m^3\,s/kg}$.

3. Solution of the initial–boundary-value problem

(a) The flow in the gap

We first solve for the horizontal velocity profile in the gap between the body and the porous free surface by integrating (2*b*) in the vertical direction and invoking the boundary conditions (4*a*, *b*) and (5*a*, *b*) to get

$$u_i = \frac{-1}{2\mu}\frac{\partial p_0}{\partial x_i}(\Delta - \eta)\eta + u_i^{+}\left(1 - \frac{\eta}{\Delta}\right) \quad (i = 1, 2), \tag{24}$$

where

$$\Delta = \int_0^t (W - v_3^-)\,dt, \quad \eta = x_3 - \int_0^t v_3^-\,dt. \tag{25a, b}$$

Here p_0 is the unknown excess water pressure in the gap and Δ is the gap thickness, which is in general a function of x_1, x_2 and t. We now investigate the relative importance of the two terms on the right-hand side of (24), i.e. the relative importance of the horizontal fluid velocity u_i^+ at the porous bed surface, compared with a typical u_i value within the gap. This can be done by a simple order-of-magnitude argument as follows. From (22) it is seen that the boundary-layer corrections to u_i^- and v_i^- are small compared with the outer solution. Therefore, from the continuity conditions (5*a*, *b*) and Hooke's law (14)

$$O(u_i^+) \sim \frac{\tau_0 a}{Gt} \quad (i = 1, 2), \tag{26}$$

where τ_0 is the horizontal shear stress exerted on the bed surface by the water flow in the gap. From (24), τ_0 is given by

$$\tau_0 = \mu \frac{\partial u_i}{\partial x_3}\bigg|_{\eta=0} = -\tfrac{1}{2}\Delta \frac{\partial p_0}{\partial x_i} - \mu \frac{u_i^+}{\Delta}. \tag{27}$$

Substituting (27) into (26), we get

$$\frac{u_i^+}{(-\Delta^2/\mu)\,\partial p_0/\partial x_i} \sim \frac{1}{1+Gt\Delta/\mu a} \quad (i = 1, 2), \tag{28}$$

which gives an order-of-magnitude estimate of the ratio between the two terms on the right-hand side of (24). A typical value of the soil shear modulus is $G \simeq 10^7\,\mathrm{N/m^2}$ (Lambe & Whitman 1979; Jaeger & Cook 1974), $\mu \simeq 10^{-3}\,\mathrm{kg/m\,s}$ and the gap horizontal dimension $a \sim O(1\,\mathrm{m})$. Substituting these values in (28), it is seen that u_i^+ may be neglected in (24), except when $\Delta \leqslant 10^{-10}\,\mathrm{m}$, where both u_i^+ and u_i are extremely small ($u_i \sim O(\Delta^2/\mu)$). Therefore we drop the u_i^+ term from (24). Then after substitution from (24) we integrate the continuity equation in the vertical direction across the gap thickness (i.e. from $\eta = 0$ to $\eta = \Delta$) and invoke the boundary conditions from (4*c*) and (5*c*) to get

$$W - nu_3^- - (1-n)\,v_3^- = \frac{1}{12\mu}\frac{\partial}{\partial x_i}\left(\Delta^3 \frac{\partial p_0}{\partial x_i}\right). \tag{29}$$

(*b*) *Boundary-layer solution in the ground*

The governing equation for p^{b} is the one-dimensional heat-diffusion equation (19),

$$K\frac{\partial^2 p^{\mathrm{b}}}{\partial x_3^2} = \frac{\partial p^{\mathrm{b}}}{\partial t}. \tag{30}$$

From (18) and (29) we have at the bed surface, underneath the body,

$$\frac{\partial p^{\mathrm{b}}}{\partial x_3} = -\frac{n}{k}(u_3^- - v_3^-) = \frac{-1}{k}\left[W - v_3^- - \frac{1}{12\mu}\frac{\partial}{\partial x_i}\left(\Delta^3\frac{\partial p_0}{\partial x_i}\right)\right] \quad \text{at} \quad x_3 = \int_0^t v_3^-\,dt. \tag{31}$$

We also require

$$p^{\mathrm{b}} \to 0 \quad \text{as} \quad x_3 \to -\infty. \tag{32}$$

The initial condition is

$$p^{\mathrm{b}} = 0 \quad \text{at} \quad t = 0. \tag{33}$$

A Laplace transform in time may be employed to solve for p^{b}, so we define the transformation

$$\bar{p}^{\mathrm{b}} = \int_0^\infty e^{-st} p^{\mathrm{b}}\,dt \tag{34}$$

and substitute into (30) to get

$$s\bar{p}^{\mathrm{b}} - K\frac{\partial^2 \bar{p}^{\mathrm{b}}}{\partial x_3^2} = 0. \tag{35}$$

The solution to (35) that satisfies (32) is

$$\bar{p}^{\mathrm{b}} = A(s)\exp\left[(s/K)^{\frac{1}{2}}x_3\right], \tag{36}$$

or

$$\bar{p}^{\mathrm{b}} = (K/s)^{\frac{1}{2}}\frac{\partial \bar{p}^{\mathrm{b}}}{\partial x_3}. \tag{37}$$

Hence, using the convolution theorem, the inverse transform gives

$$p^{\mathrm{b}} = \int_0^t g(t-\tau)\frac{\partial p^{\mathrm{b}}}{\partial x_3}(x_1, x_2, x_3, \tau)\,d\tau, \tag{38}$$

where $g(t)$ is the inverse transform of $(K/s)^{\frac{1}{2}}$, i.e.

$$g(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}(K/s)^{\frac{1}{2}}\,ds = (K/\pi t)^{\frac{1}{2}}. \tag{39}$$

At the bed surface $\eta = 0$, we invoke the condition (9) for the fluid pressure continuity and substitute from (31) to get

$$p_0^{\mathrm{b}} = \frac{-(K/\pi)^{\frac{1}{2}}}{k}\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\left[W - v_3^- - \frac{1}{12\mu}\frac{\partial}{\partial x_i}\left(\Delta^3\frac{\partial p_0}{\partial x_i}\right)\right] \tag{40}$$

where

$$p_0 = p_0^{\mathrm{b}} + p_0^{\mathrm{o}}, \tag{41}$$

with p_0 being the pressure in the gap, p_0^{o} and p_0^{b} being respectively the outer and the boundary-layer parts of the solution for the pore pressure at the bed surface.

(c) *The outer solution in the ground*

The governing equations for the outer stresses τ_{ij}^{o} and p^{o} are (12) and (17). Since, from (21), τ_{3i}^{b} are negligible, we may impose the stress-continuity conditions ($8a, b, c$) on τ_{3i}^{o}. Furthermore, these conditions will be applied at the initial undeformed location of the free surface, $x_3 = 0$, instead of the current deformed surface, with an error $O(\int_0^t v_3^- \,dt/a)$. This is anticipated to be within the lubrication-theory approximation ($\Delta/a \ll 1$). Therefore the outer problem is stated as follows:

$$\frac{\partial \tau_{ij}^{\mathrm{o}}}{\partial x_j} = 0 \quad (x_3 \leqslant 0), \tag{42a}$$

$$\tau_{3i}^{\mathrm{o}} = -p_0\delta_{3i} + \mu\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right)^+ \quad (x_3 = 0), \tag{42b}$$

$$p^{\mathrm{o}} = -\frac{(1-2\nu_{\mathrm{e}})\beta}{2(1+\nu_{\mathrm{e}})nG}\tau_{kk}^{\mathrm{o}}. \tag{42c}$$

Since the boundary value in ($42b$) (or the forcing at the free surface) is still unknown, we proceed first to investigate some aspects of the above elastostatic problem, with general unspecified forcing τ_{3i}^{o}.

Because the problem is linear, we may decompose it into a number of simpler ones and then use superposition to reconstruct the solution.

(1) We first consider the problem of general pressure stresses applied on an arbitrary surface area, i.e.

$$\frac{\partial \tau_{ij}^{(1)}}{\partial x_j} = 0,$$

with

$$\left.\begin{aligned} \tau_{33}^{(1)} &= \tau_{33}^{(1)}(x_1, x_2, t), \\ \tau_{31}^{(1)} &= \tau_{32}^{(1)} = 0, \end{aligned}\right\} \quad x_3 = 0.$$

Substitution of the constitutive equations (14) into the equilibrium equations yields

$$G\left[\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{1}{1-2\nu_e}\frac{\partial^2 v_j}{\partial x_i \partial x_j}\right] = 0. \tag{43}$$

Taking the divergence of (43), one gets an equation for the dilatation c:

$$\nabla^2 c = 0, \quad c = \frac{\partial v_k}{\partial x_k}.$$

The problem may be solved formally using the two-dimensional Fourier transform in the horizontal plane (x_1, x_2):

$$\bar{c} = \iint_{-\infty}^{\infty} e^{i\boldsymbol{\lambda}.\mathbf{x}}\, c\, d\mathbf{x},$$

where $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, the Fourier variables, and the overbar is used to denote the Fourier transform. The solution for $\bar{c}$ in the lower half-space is formally

$$\bar{c} = \bar{c}_0 e^{\lambda x_3}, \quad \lambda^2 = \lambda_1^2 + \lambda_2^2. \tag{44}$$

Using the boundary conditions of zero shear stresses at the surface it can be shown that

$$\left.\frac{\partial \bar{c}}{\partial x_3}\right|_{x_3=0} = \left[\lambda^2 \bar{v}_3 + \frac{\partial^2 \bar{v}_3}{\partial x_3^2}\right]_{x_3=0}. \tag{45}$$

However, the Fourier transform of the vertical component of (43) gives

$$\frac{\partial^2 \bar{v}_3}{\partial x_3} - \lambda^2 \bar{v}_3 + \frac{1}{1-2\nu_e}\frac{\partial \bar{c}}{\partial x_3} = 0,$$

which has the solution

$$\bar{v}_3 = A(\lambda)\, e^{\lambda x_3} - \frac{\bar{c}_0}{2(1-2\nu_e)}\, x_3 e^{\lambda x_3}. \tag{46a}$$

Substitution of (44) and (46a) into (45) determines A:

$$A = \frac{1-\nu_e}{\lambda(1-2\nu_e)}\,\bar{c}_0. \tag{46b}$$

From (14) we get the Fourier transform of $\tau_{33}^{(1)}$:

$$\frac{\partial}{\partial t}\bar{\tau}_{33}^{(1)} = 2G\left[\frac{\partial \bar{v}_3}{\partial x_3} + \frac{\nu_e}{1-2\nu_e}\,\bar{c}\right],$$

which upon substitution from (44) and (46) gives

$$\left.\frac{\partial}{\partial t}\bar{\tau}_{33}^{(1)}\right|_{x_3=0} = \frac{G}{1-2\nu_e}\,\bar{c}_0.$$

The inverse transform of the above equation is simply

$$\frac{\partial}{\partial t}\tau_{33}^{(1)}\bigg|_{x_3=0} = \frac{G}{1-2\nu_e}c_0, \quad c_0 = \frac{\partial v_k}{\partial x_k}\bigg|_{x_3=0},$$

or, expressing c_0 in terms of the trace of the stress tensor $\tau_{kk}^{(1)}$, by contracting (14), we get the general and important result

$$\tau_{kk}^{(1)} = 2(1-\nu_e)\tau_{33}^{(1)}, \quad x_3 = 0, \tag{47a}$$

for any $\tau_{33}^{(1)}(x_1, x_2, 0, t)$.

(2) The second problem we consider is the stress field in the semi-infinite space due to the viscous shear stress applied at the surface as water flows laterally into the gap. However, the order of magnitude of this surface shear stress τ_0 can be shown to be Δ/a times the dynamic pressure stress in the gap. This classical lubrication-theory result (see e.g. Lamb 1932, p. 583) can be seen immediately from (2*b*) which essentially balance the vertical gradient of shear stress ($\sim \tau_0/\Delta$) with the horizontal gradient of pressure ($\sim p_0/a$). Therefore, dilatation due to surface shear may be neglected compared with dilatation due to surface pressure, and for this problem we take

$$\frac{\partial}{\partial x_k}v_k^{(2)} = \tau_{kk}^{(2)} \simeq 0, \quad x_3 = 0. \tag{47b}$$

Now we return to the original problem. At the free surface $x_3 = 0$ the outer pore pressure p_0^o is given by (17), which in view of (47*a*, *b*) reduces to

$$p_0^o = -\frac{(1-2\nu_e)}{nG}\beta\tau_{33}^o, \quad x_3 = 0. \tag{48}$$

Then, since from (2*a*) and (24) we have

$$\mu\frac{\partial u_3}{\partial x_3} \sim p_0\left(\frac{\Delta}{a}\right)^2 \ll p_0,$$

the third of the boundary conditions (42*b*) gives

$$\tau_{33}^o = -p_0 = -(p_0^o + p_0^b), \quad x_3 = 0; \tag{49}$$

so that from (48) we get the following general result for any free-surface pressure loading:

$$p_0^b = \frac{m}{1+m}p_0, \quad m = \frac{nG}{(1-2\nu)\beta}, \tag{50}$$

where (15) has been used to express ν_e in terms of the solid-matrix Poisson's ratio ν. The parameter m is essentially the ratio G/β of the skeleton to the fluid elasticity, or simply the stiffness ratio between the two constituents. Substituting (50) into (40), we finally obtain the governing equation for the pressure in the gap:

$$p_0 = -\alpha\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\left[\frac{\partial\Delta}{\partial\tau} - \frac{1}{12\mu}\frac{\partial}{\partial x_i}\left(\Delta^3\frac{\partial p_0}{\partial x_i}\right)\right], \tag{51a}$$

where, after substitution from (19) for K,

$$\alpha = \frac{1+m}{m}\left[\frac{G/k}{\pi(1-2\nu)m + \dfrac{\pi(1-2\nu)}{2(1-\nu)}}\right]^{\frac{1}{2}}, \tag{51b}$$

and where, from (25*a*),

$$\frac{\partial \Delta}{\partial t} = W - v_3^-. \tag{52}$$

Equation (51) is an integro-differential equation for two unknowns p_0 and the gap thickness Δ (or the vertical (swelling) velocity v_3^- of the porous free surface when the body velocity W is prescribed). From (22) it is seen that v_3^- is to be obtained from the outer elasticity problem. Therefore, the initial–boundary-value problem is reduced to solving (51) for the pressure underneath the ascending body, coupled with the elastostatic problem ((14), (42*a*, *b*)).

4. Approximate analytical solution for the breakout force–time relation

The coupling between (51) and the elastostatic problem in the ground is, however, still difficult to deal with analytically. Even numerical manipulation seems tedious and quite involved, since for one thing the system is a transient coupled system. However, one may get rid of this coupling by solving the problem in a partially reverse order. First, replace Δ by its spatial average $\bar{\Delta}(t)$, i.e. replace (52) by its spatial average

$$\frac{d\bar{\Delta}}{dt} = W - (v_3^-)_{\mathrm{av}}, \tag{53}$$

where $(v_3^-)_{\mathrm{av}}$ is the average vertical velocity of the bed surface underneath the body, so that (51*a*) becomes

$$p_0 = -\alpha \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \left[\frac{d\bar{\Delta}}{d\tau} - \frac{\bar{\Delta}^3}{12\mu} \nabla^2 p_0\right]. \tag{54a}$$

Now, one may proceed to solve (54*a*) for p_0 in terms of $\bar{\Delta}(t)$. The absolute velocity $W(t)$ might be obtained afterwards by solving the elastostatic problem, with p_0 given, to get $(v_3^-)_{\mathrm{av}}$ and then using (53) to get W. The solution obtained for p_0 will, however, be based on neglecting the spatial variation of Δ (i.e. of v_3^-) underneath the ascending body. Therefore it should be considered as a leading-order solution in the 'small' parameter

$$\sigma = \frac{1}{A} \int_A (|V_3^- - (V_3^-)_{\mathrm{av}}|/(V_3^-)_{\mathrm{av}})\, dA \ll 1 \quad \text{for all} \quad t > 0, \tag{54b}$$

where

$$V_3^- = \int_0^t v_3^-\, dt, \quad (V_3^-)_{\mathrm{av}} = \int_0^t (v_3^-)_{\mathrm{av}}\, dt \tag{54c}$$

are respectively the vertical deflection of the free surface and its spatial average over the base area A. The validity of such a solution will therefore be based on having σ much less than unity during the whole process of breakout. The smallness of σ is evident in many elasticity solutions in the literature for different finite loading areas (see e.g. Ahlvin & Ulery 1962; Deresiewicz 1960; Poulos & Davis 1974, p. 43) and different loading distributions (e.g. Poulos & Davis 1974). For example, the tabulated solution reported by Ahlvin & Ulery (1962) shows $\sigma = 0{\cdot}102$ for uniform circular loading. Refinement to such a leading-order solution for p_0 will be discussed later, but what will be attempted now is solving this leading-order uncoupled problem; i.e. the relative deflections of the sea bed underneath the body will be ignored so that one may proceed to solve the integro-differential equation (54*a*).

(*a*) *Breakout of axisymmetric bodies*

As a first example, we consider the breakout of axisymmetric bodies, where the shape of the body base is circular with radius a. Here the polar co-ordinates (r, θ) may be used instead of (x_1, x_2) and, since the problem is axisymmetric, the pressure p_0 will be a function of the radial co-ordinate $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ but not of the angle $\theta = \arctan (x_1/x_2)$. Therefore, (54*a*) reduces to

$$p_0 = -\alpha \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \left[\frac{d\bar{\Delta}}{d\tau} - \frac{\bar{\Delta}^3}{12\mu} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p_0}{\partial r} \right) \right]. \tag{55}$$

The total pressure force F_p on the body is given by

$$F_p = \int_0^a 2\pi r p_0 \, dr = -\pi a^2 \alpha \int_0^t \left(\frac{d\bar{\Delta}}{d\tau} \right) \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} + \frac{2\pi a \alpha}{12\mu} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \bar{\Delta}^3 \frac{\partial p_0}{\partial r} \bigg|_{r=a}. \tag{56}$$

Notice that, beside $\bar{\Delta}$, the pressure gradient $\partial p_0 / \partial r$ at the edge of the gap $(r = a)$ is needed in order to get the total pressure force F_p. Therefore, we shall investigate the solution for p_0 near the edge, $r \simeq a$. Expanding p_0 in a Taylor series in time,

$$p_0(r, \tau) = p_0(r, t) + (\tau - t) \frac{\partial p_0}{\partial t}(r, t) + \tfrac{1}{2}(\tau - t)^2 \frac{\partial^2 p_0}{\partial t^2}(r, t) + \ldots, \tag{57}$$

and, substituting into (55), one gets

$$p_0(r, t) = -\alpha \left\{ G - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p_0}{\partial r} \right) \int_0^t \frac{\bar{\Delta}^3}{12\mu (t-\tau)^{\frac{1}{2}}} d\tau + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^2 p_0}{\partial r \, \partial t} \right) \int_0^t \frac{\bar{\Delta}^3}{12\mu} (t-\tau)^{\frac{1}{2}} d\tau \right.$$
$$\left. - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^3 p_0}{\partial r \, \partial t^2} \right) \int_0^t \frac{\bar{\Delta}^3}{24\mu} (t-\tau)^{\frac{3}{2}} d\tau + \ldots \right\}, \tag{58a}$$

where

$$G = \int_0^t \left(\frac{d\bar{\Delta}}{d\tau} \right) \frac{d\tau}{(t-\tau)^{\frac{1}{2}}}. \tag{58b}$$

Equation (58*a*) is to be solved subject to the boundary condition from (3),

$$p_0 = 0 \quad \text{at} \quad r = a; \tag{59}$$

we also require p_0 to be finite for $r \leqslant a$. Notice first that, if only the first term in the Taylor expansion (57) is kept and (58*a*) is solved, ignoring

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^2 p_0}{\partial r \, \partial t} \right), \quad \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial^3 p_0}{\partial r \, \partial t^2} \right)$$

etc., the solution would simply be

$$p_0 = p_0^{(1)} = -\alpha G \left[1 - \frac{I_0(fr)}{I_0(fa)} \right], \tag{60}$$

where $I_0(fr)$ is the modified Bessel function of the zeroth order, with f being given by

$$f = \left[\alpha \int_0^t \frac{\bar{\Delta}^3}{12\mu (t-\tau)^{\frac{1}{2}}} d\tau \right]^{-\frac{1}{2}}.$$

Therefore, keeping all the terms in (58*a*), one may try the expansion

$$p_0 = -\alpha G\left[1-\frac{I_0(f_0 r)}{I_0(f_0 a)}\right]+(a-r)^3 f_1(t)+(a-r)^4 f_2(t)+\ldots, \tag{61}$$

so that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_0}{\partial r}\right) = \alpha G f_0^2 \frac{I_0(f_0 r)}{I_0(f_0 a)}+[6(a-r)-(3/r)(a-r)^2]f_1(t)$$
$$+[12(a-r)^2-(4/r)(a-r)^3]f_2(t)+\ldots, \tag{62}$$

where $f_0, f_1, f_2 \ldots$ are functions of time only. But, for $p_0 = 0$ at $r = a$ one must have from (55)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{p_0}{r}\right)_{r=a} = \alpha G f_0^2 = \frac{12\mu}{\bar{\Delta}^3}\frac{d\bar{\Delta}}{dt}, \tag{63}$$

which determines the function $f_0(t)$ in terms of $\bar{\Delta}(t)$. It is easy to see that the function f_0 is different from the function f in (60). The rest of the functions $f_1(t), f_2(t)$, etc. in (61) can be found by substituting (61), (62) and its time derivatives into (58) and equating coefficients of equal powers of $(a-r)^q$ ($q = 1, 2, \ldots$). It is seen then that the expansion (61) will provide a unique solution for all the time functions $f_0(t)$, $f_i(t)$, and the series will be convergent in a region sufficiently close to the periphery of the gap. However, at the boundary $r = a$, the pressure gradient is given by

$$\frac{\partial p_0}{\partial r} = \alpha G f_0 \frac{I_1(f_0 a)}{I_0(f_0 a)} \quad \text{at} \quad r = a; \tag{64}$$

i.e. it is independent of f_1, f_2, etc. Therefore, the substitution of (64) into (56), and the use of (63), leads to

$$F_p = -\pi a^2 \alpha \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\left(\frac{d\bar{\Delta}}{d\tau}\right)\left[1-\frac{2I_1(f_0 a)}{f_0 a\, I_0(f_0 a)}\right], \tag{65}$$

which finally gives the *exact* solution of (56) for the total pressure force $F_p(t)$ in terms of the average gap thickness $\bar{\Delta}(t)$. Note that (65) is *Abel's integral equation* (see e.g. Carrier, Krook & Pearson 1966, p. 357) which can be inverted to give

$$\frac{d\bar{\Delta}}{dt}\left[1-\frac{2I_1(f_0 a)}{(f_0 a)\, I_0(f_0 a)}\right] = \frac{-1}{\pi^2 a^2 \alpha}\frac{d}{dt}\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} F_p(\tau). \tag{66}$$

This is a more useful formula when the input to the problem is the time history of the pulling force $|F_p(t)|$.

(*b*) *Breakout of two-dimensional bodies*

As a second example, we consider the breakout of a slender body, whose length is much longer than its beam (as is the case for most ships for example). The problem is therefore reduced locally to a two-dimensional one. Aligning the x_2 axis along the longitudinal axis of the body base and taking the base width in the x_1 direction to be $2a$, (54*a*) will then reduce to

$$p_0 = -\alpha \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}}\left[\frac{d\bar{\Delta}}{d\tau}-\frac{\bar{\Delta}^3}{12\mu}\frac{\partial^2 p_0}{\partial x_1^2}\right] \quad (-a \leqslant x_1 \leqslant a). \tag{67}$$

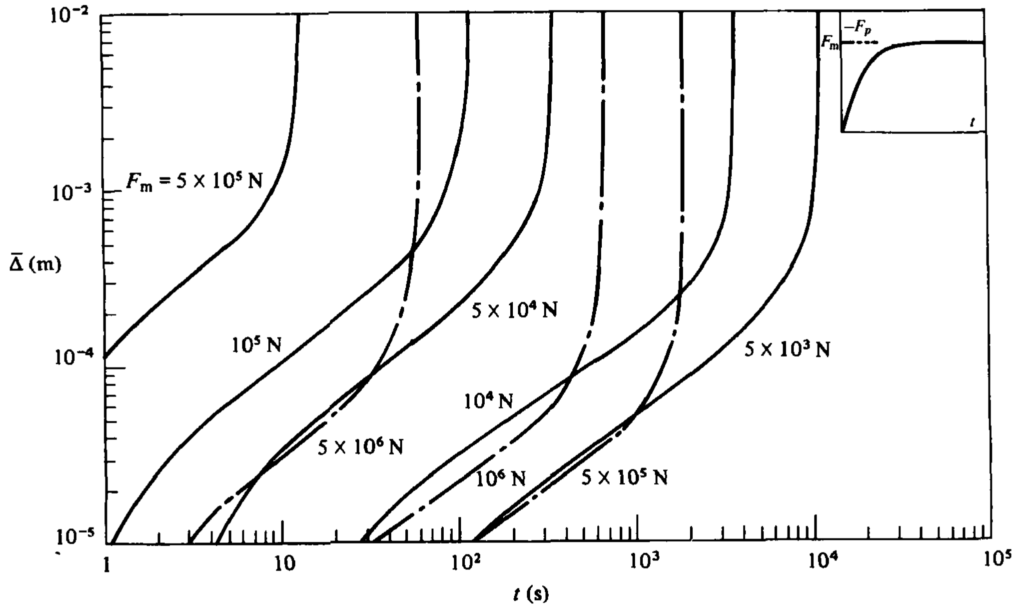

FIGURE 2. The gap-expansion history $\bar{\Delta}(t)$ under a two-dimensional body of beam $2a = 10$ m due to an applied pulling force $|F_p(t)|$ as shown in the inset. The sea bed has permeability coefficient $k = 10^{-9}$ m^3 s/kg and shear modulus $G = 10^7$ N/m^2 (sandy soil); $\cdot - \cdot$, complete saturation, $\beta = 1{\cdot}9 \times 10^9$ N/m^2; ——, saturation slightly less than unity, $\beta = 10^7$ N/m^2.

Following the same procedure as that described above for the axisymmetric bodies, it is straightforward to show that the total pressure force F_p per unit length of the body is given similarly by

$$F_p = -2a\alpha \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \left(\frac{d\bar{\Delta}}{d\tau}\right) \left[1 - \frac{\tanh(f_0 a)}{f_0 a}\right], \tag{68}$$

where f_0 is still given by (63),

$$\alpha G f_0^2 = \frac{12\mu}{\bar{\Delta}^3} \frac{d\bar{\Delta}}{d\tau},$$

and G given by (58*b*).

Again, (68) is Abel's integral equation, which when inverted gives

$$\frac{d\bar{\Delta}}{dt}\left(1 - \frac{\tanh(f_0 a)}{f_0 a}\right) = \frac{-1}{2\pi a\alpha} \frac{d}{dt} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} F_p(\tau). \tag{69}$$

(*c*) *Discussion of results*

Many of the essential features of the breakout phenomenon can now be brought out by discussing the special solutions: (65) and (68), which describe how the negative pressure or suction force $F_p(t)$ develops underneath axisymmetric and slender bodies respectively as they detach themselves from the sea bottom; and (66) and (69), which describe how the gap $\bar{\Delta}(t)$ underneath the body expands with time owing to the prescribed pulling force $|F_p(t)|$. These solutions are only applicable within the time limit imposed by (23), and are based on ignoring inertia effects and the horizontal variation of the bed-surface deformation under the ascending body. The limitations imposed on the theory because of these assumptions will be discussed here as well.

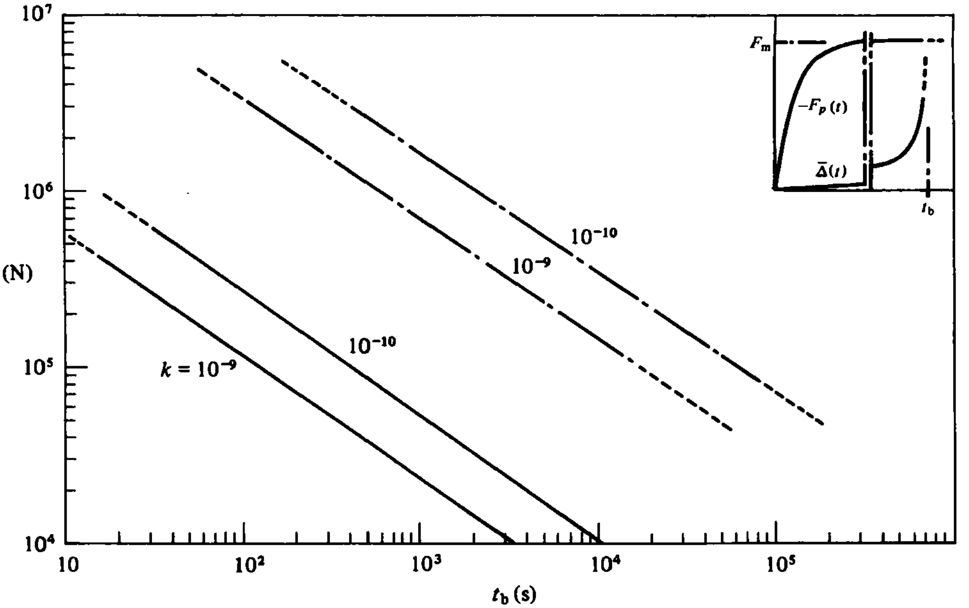

FIGURE 3. The breakout force–time relations $F_m(t_b)$ for a two-dimensional body of beam $2a = 10$ m for two different soil-permeability coefficients $k = 10^{-9}$ and 10^{-10} m^3 s/kg, with G fixed at 10^7 N/m^2. · — ·, complete saturation, $\beta = 1\cdot 9 \times 10^9$ N/m^2; ——, saturation slightly less than unity, $\beta = 10^7$ N/m^2.

Figure 2 shows the results of the numerical integration of (69) for $\bar{\Delta}(t)$ under a two-dimensional body of constant beam $2a = 10$ m with the model pulling-force history

$$-F_p(t) = \frac{t}{b+t} F_m, \quad b = 1, \tag{70}$$

which gives for the right-hand side of (69)

$$\frac{d}{dt}\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} F_p(\tau) = -F_m\left[\frac{b}{2(b+t)^{\frac{3}{2}}}\ln\frac{1+(t/[b+t])^{\frac{1}{2}}}{1-(t/[b+t])^{\frac{1}{2}}} + \frac{t^{\frac{1}{2}}}{b+t}\right], \tag{71}$$

and where F_m is the maximum constant value of the pulling force attained asymptotically as $t \to \infty$. A sketch of this model force history is shown in the inset of figure 2. Figure 2 shows $\bar{\Delta}(t)$ for different values of F_m and selected soil parameters. The interesting feature in all the curves is that $\bar{\Delta}$ remains small ($\leqslant 10^{-3}$ m) for most of the pull-up process and then there is a rather sharp increase in the rate of expansion $d\bar{\Delta}/dt \to \infty$ until the curve becomes vertical at the '*breakout time*' t_b. This feature of sharp increase in the rate of expansion, as if the soil has suddenly lost its holding strength to the ascending body, has been noticed and reported repeatedly in many field and laboratory experiments (e.g. Liu 1969). Examining (63), it is seen that, as $t \to \infty$, $f_0 \to 0$ so that in (69) $\tanh(f_0 a)/f_0 a \to 1$, corresponding to large $d\bar{\Delta}/dt$. Therefore, the condition

$$f_0 a \ll 1, \tag{72}$$

may be considered as the criterion that defines the breakout time t_b. Notice from (66) that this criterion applies as well in the case of axisymmetric bodies. Each $\bar{\Delta}(t)$ curve in figure 2 corresponds to a specific pair of breakout force, defined here as F_m, and breakout time t_b, and hence a point on the desired breakout force–time relationships. Some of these relationships are shown in figure 3 for selected soil parameters (soil permeability k, soil firmness G and pore-fluid bulk modulus β). In all the calculations carried out here Poisson's ratio ν and porosity n are kept constant and equal to $\frac{1}{3}$ and 0·3 respectively. Figure 3 shows the results $F_m(t_b)$ for two different soil properties under the same slender body of beam $2a = 10$ m. One is for sandy soil with $G = 10^7\,\mathrm{N/m^2}$ and $k = 10^{-9}\,\mathrm{m^3\,s/kg}$, and the other is for soil of fine-sand–silt mixture with $G = 10^7\,\mathrm{N/m^2}$ and $k = 10^{-10}\,\mathrm{m^3\,s/kg}$. The results are shown for the case of complete pore-saturation ($\beta = \beta_0 = 1{\cdot}9\times10^9\,\mathrm{N/m^2}$) (broken lines), as well as for partial saturation ($\beta = 10^7\,\mathrm{N/m^2}$) (solid lines). All four breakout force–time relations that are plotted on a log–log scale in figure 3 show approximately the same functional dependence

$$F_m = At_b^{-B}, \tag{73}$$

with the exponent B having an approximately constant value equal to 0·7. These relations are limited in extent at both ends because of the theory's limiting assumptions (shown in the figure by dotted extensions of the lines). Near the longer-breakout-time end the limiting factor is the increase of the non-dimensional time scale T, from (23)

$$T = \epsilon^2 = \frac{Kt}{a^2} \ll 1, \tag{74}$$

and hence the breakdown of the boundary-layer formulation. An important point to notice, however, is that, although the breakout time t_b reaches an hour or longer, the non-dimensional time T remains smaller than unity for a wide range of parameters satisfying (74) and in accordance with the short-time boundary-layer formulation. Near the shorter-breakout-time end, the limiting factor is the increase of Reynolds number defined by (1). The largest Reynolds numbers will occur right at the gap edge, where the horizontal velocities are maximum. Use of (24) and (64) shows that condition (1) is satisfied throughout the breakout process (except of course at the very late stages near t_b, where $\bar{\Delta}(t)$ is near vertical) for a wide range of F_m values. The inertia effects on the other hand will affect the results at the later stages, especially for larger F_m, by acting to resist the water inflow to the gap from the sides, and hence somewhat delaying the breakout time from the present inertia-free calculations. There is also another limit to the results that was pointed out by one of this paper's referees, namely when the water goes into pure tension. This would impose a limit on the higher breakout forces F_m, especially in shallow water. From figure 3 it is also seen that, for a given breakout force, the breakout time increases with decreasing permeability as may be anticipated intuitively. There is also a significant reduction in the breakout time for a given breakout force when the pore saturation ratio is reduced slightly below unity (thus reducing the bulk modulus β to $10^7\,\mathrm{N/m^2}$).

It is furthermore interesting to study the case when $\bar{\Delta}(t)$ is given as an input to the problem, i.e. to solve (65) or (68) for the development of $F_p(t)$ due to a prescribed $\bar{\Delta}(t)$. Figure 4 shows sample results for the pressure-force history $F_p(t)$ under an axisymmetric body of base radius $a = 1$ m and for selected soil parameters, with the model gap-expansion function

$$\bar{\Delta} = \gamma t, \tag{75}$$

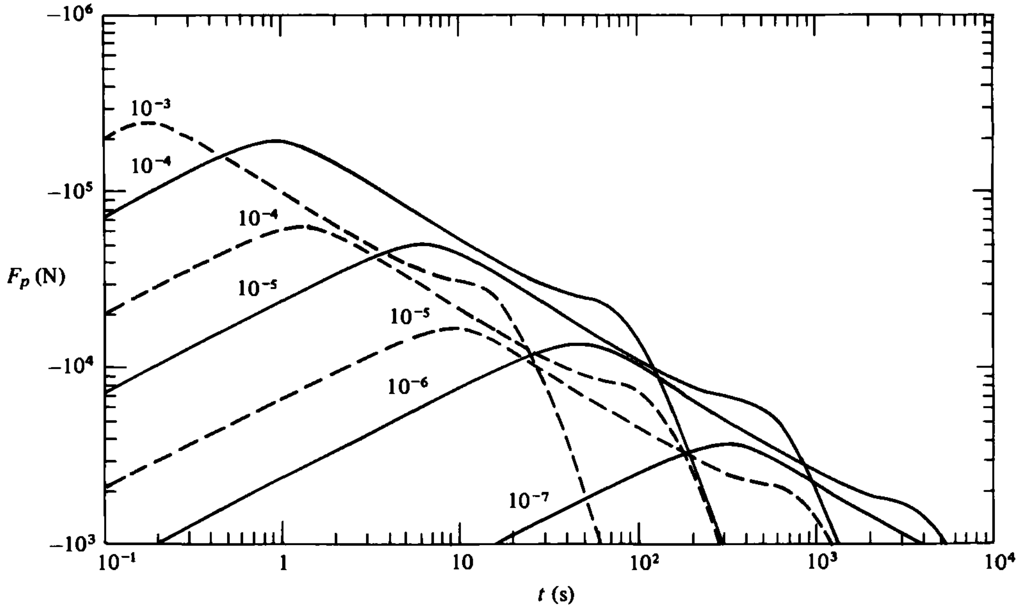

FIGURE 4. Negative-pressure-force history $F_p(t)$ developed underneath an axisymmetric body of base radius $a = 1$ m as it detaches itself from a sea bottom of permeability $k = 10^{-10}$ m^3 s/kg (fine-sand-silt mixture) and pore-fluid bulk modulus $\beta = 10^6$ N/m^2 (saturation slightly below unity). ——, shear modulus $G = 10^7$ N/m^2; - - -, $G = 10^6$ N/m^2. Labels denote gap-expansion rate γ in m/s.

where γ is the constant rate of expansion for this linear model. Substituting (75) into (58*b*) and (63), we get (for either the axisymmetric or the two-dimensional case)

$$G = 2\gamma t^{\frac{1}{2}}, \quad f_0 = \left(\frac{6\mu}{\alpha\gamma^3}\right)^{\frac{1}{4}} t^{-\frac{3}{4}}. \tag{76}$$

Notice also that $f_0 \to 0$ as $t \to \infty$ and the condition (72) corresponds to the asymptotic vanishing of the integrand in either (65) or (68). Each curve in figure 4 corresponds to a different expansion velocity γ. Although the breakout mechanism can be explained from figure 2 for $\bar{\Delta}(t)$, figure 4 helps more in understanding the details of this mechanism. The general behaviour of the $F_p(t)$ curves in figure 4 is seen to be first an initial build-up, from zero initial value, of negative pressure force; proportional to $t^{\frac{1}{2}}$ where the first term in the integrand in (55) dominates over the second one. This means that during this initial stage most of the water supply to the gap is coming vertically from the underlying porous bed, with very little horizontal water influx from the gap periphery. The increasing difference between the pressure in the gap and the ambient water, along with the increasing gap thickness, will then force more water to flow into the gap through its perimeter and hence reduce the amount of water pumped out of the porous bed. This relieving action manifests itself in figure 4 by the subsequent levelling-off of the $F_p(t)$ curves until they reach a maximum $|F_p| = \tilde{F}_m$. Further increase in the gap thickness will then result in a decay in F_p. Along the descending limbs of the $F_p(t)$ curves, it is the second term in the integrand in (55) which dominates the first one, meaning that most of the water supply into the gap is now derived by the horizontal pressure gradient in the gap and coming from the sides with diminishing supply from the ground. Again the interesting feature in all the curves in figure 4 is that the decay rate on the descending limb is not uniform. Instead, there is a distinct interval on the curve where it experiences a sharp and a rather sudden increase in the decay rate (e.g. from $F_p \sim t^{-0.6}$ to $F_p \sim t^{-2.6}$), and hence a sudden drop in the soil 'holding strength' against the body breakout. This interval always corresponds to small $f_0 a$ ($f_0 a \simeq 0.002$–0.004) in accordance with condition (72) and corresponding to 'breakout' conditions.

One may recall here that in all the situations considered in this paper the object is pulled up uniformly without any tilt. It is expected that, if a finite tilt $\theta(t)$ is added to the uniform body velocity $W(t)$, the developed negative pressure in the gap will be redistributed over the horizontal gap area in order to provide for the deviation of the gap thickness from that in the case of uniform pull-up. Furthermore, the supply of water flowing through the periphery of the gap, which represents the relieving action responsible for the body breakout, will not be uniform along the periphery. This means that 'local breakout conditions' might be reached over a certain portion of the gap area, while the rest is still attached. The region of local breakout will then propagate with time (in a similar way to crack propagation in solids for example) until covering the whole area, corresponding to total breakout. One might expect that such non-uniform breakout would be somewhat faster than the uniform 'without tilt' breakout case. Investigation of this point is being pursued further at the present.

One final remark concerns the effect of neglecting the horizontal variation of the bed-surface deformation under the ascending body. An improvement may be attempted by using the solution obtained for p_0 (cf. (61)) and solving the outer elasticity problem to find the surface deformation, which in turn can be used to get an improved solution for (51). This may be repeated several times to improve on the accuracy of the results. This iterative scheme seems a lot easier to handle than solving the coupled equations.

The present theory, although it involves a number of assumptions, is shown to be applicable within a wide range of the relevant physical parameters and is also shown to capture the essential features of the breakout phenomenon. In particular, it shows the important dependence of the breakout process on the soil permeability k and the soil firmness G. It also shows the effect of the pore-fluid compressibility, which is greatly affected by the presence of tiny air bubbles, due in part to organic processes in the sea bottom. Moreover, it reproduces the important feature commonly observed in salvage operations and laboratory experiments of the rather sharp drop in the ocean-sediment holding strength when reaching the breakout time.

The author is indebted to Professor C. C. Mei of M.I.T. for the many valuable discussions that helped tremendously in bringing this paper into its present form. In particular, the idea of inverting (65) and (68) into (66) and (69) respectively, should be credited to him. The author wishes also to thank Professor D. R. F. Harleman of M.I.T. who suggested the problem, and whose experimental data actually motivated this study.

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Appendix. Reduction of Biot's quasistatic consolidation theory to the boundary-layer formulation

Terzaghi's one-dimensional soil consolidation theory was extended by Biot (1941) to three dimensions. In Biot (1941), hereinafter referred to as B 41, inertia effects are neglected and the governing equations for the two-phase soil system are given in the notation of this paper by:

(i) stress equilibrium conditions ((1.2) in B 41)

$$\frac{\partial \tau_{ij}}{\partial x_j} = 0; \tag{A 1}$$

(ii) Darcy's law (from (4.2) and (4.3) in B 41)

$$-k\frac{\partial p}{\partial x_i} = n(u_i - v_i); \tag{A 2}$$

(iii) continuity ((2.12) in B 41)

$$n\frac{\partial u_i}{\partial x_i} + (\hat{\alpha} - n)\frac{\partial v_i}{\partial x_i} = \frac{-1}{Q}\frac{\partial p}{\partial t}; \tag{A 3}$$

(iv) stress–strain constitutive equations ((2.11) in B 41)

$$\frac{\partial}{\partial t}\tau_{ij} = G\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} + \frac{2\nu}{1-2\nu}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right] - \hat{\alpha}\frac{\partial p}{\partial t}\delta_{ij}. \tag{A 4}$$

Now, assuming that the solid grains are incompressible, i.e., the solid-grain density $\rho_s =$ constant, it can be shown (Biot & Willis 1957) that the material constants $\hat{\alpha}$ and Q are given by

$$\hat{\alpha} = 1, \quad Q = \beta/n. \tag{A 5}$$

From the constitutive equations (A 4), dimensional arguments show that the velocities

$$(v_i, u_i) \sim O\left(\frac{Pa}{Gt}\right), \tag{A 6}$$

where P is a characteristic stress, i.e. $\tau_{ij}, p \sim O(P)$ and a is the body size. Substituting these orders of magnitude into Darcy's law (A 2), it is seen that the leading-order balance of momentum gives

$$u_i - v_i \sim O\left(\frac{Gkt}{a^2}\right). \tag{A 7}$$

Thus, for

$$\frac{Gkt}{a^2} \ll 1, \tag{A 8}$$

which is essentially condition (23), we have, to leading order,

$$u_i = v_i. \tag{A 9}$$

Substituting (A 9) into the continuity equation (A 3), (13) follows. Substituting (13) into the constitutive equation (A 4), (14) follows directly. Now, near the free surface, appreciable fluid motion relative to the solid can take place, and hence (A 9) and the implied approximation above break down in a thin boundary layer of thickness δ

near the free surface. Moreover, following the above approximation, the problem is essentially reduced to a conventional elastostatic problem (*the outer problem*), which is of lower order than Biot's original equations. Hence, it cannot, in general, satisfy all the prescribed boundary conditions of the original problem. A *boundary-layer correction* should therefore be added near the boundaries in order to satisfy the remaining boundary conditions. This implies that the boundary-layer correction and the outer solution will be, in general, of the same order of magnitude near the boundaries.

In the boundary-layer solution, δ should be the characteristic length scale in the vertical direction x_3 instead of a. This immediately implies from the equilibrium equations (A 1) that

$$\tau^{\mathrm{b}}_{i3} = 0, \tag{A 10}$$

to leading order. On the other hand, substituting (A 4) into (A 1) yields, in vector notation

$$G\left(\nabla^2\mathbf{v}^{\mathrm{b}}+\frac{1}{1-2\nu}\nabla\nabla.\mathbf{v}^{\mathrm{b}}\right)-\nabla\frac{\partial p^{\mathrm{b}}}{\partial t} = 0. \tag{A 11}$$

Taking the curl of (A 11), one easily gets

$$\nabla^2(\nabla\times\mathbf{v}^{\mathrm{b}}) \simeq \frac{\partial}{\partial x_3^2}(\nabla\times\mathbf{\dot{v}}^{\mathrm{b}}) = 0. \tag{A 12}$$

The last step is because $\partial^2(\;)^{\mathrm{b}}/\partial x_3^2 \gg \partial^2(\;)^{\mathrm{b}}/\partial x_1^2,\ \partial^2(\;)^{\mathrm{b}}/\partial x_2^2$. However, since $\nabla\times\mathbf{v}^{\mathrm{b}}$ vanishes identically outside the boundary layer $|x_3| \gg \delta$, it must be zero throughout, i.e.

$$\nabla\times\mathbf{v}^{\mathrm{b}} = 0. \tag{A 13}$$

Thus, the boundary-layer correction of the solid velocity is irrotational, which in turn implies that the horizontal velocities are much less than the vertical

$$v_i^{\mathrm{b}}/v_3^{\mathrm{b}} = O(\delta/a) \quad (i = 1,2). \tag{A 14}$$

Using this fact, the dominant part of (A 11) becomes

$$G\left(\frac{\partial^2 v_i^{\mathrm{b}}}{\partial x_3^2}+\frac{1}{1-2\nu}\frac{\partial^2 v_3^{\mathrm{b}}}{\partial x_i\,\partial x_3}\right)-\frac{\partial^2 p^{\mathrm{b}}}{\partial x_i\,\partial t} = 0 \quad (i = 1,2), \tag{A 15a}$$

$$G\left(\frac{\partial^2 v_3^{\mathrm{b}}}{\partial x_3^2}+\frac{1}{1-2\nu}\frac{\partial^2 v_3^{\mathrm{b}}}{\partial x_3^2}\right)-\frac{\partial^2 p^{\mathrm{b}}}{\partial x_3\,\partial t} = 0. \tag{A 15b}$$

Substituting the continuity equation (A 3) into Darcy's law (A 2), one gets for the boundary-layer correction

$$k\frac{\partial^2 p^{\mathrm{b}}}{\partial x_j\,\partial x_j} = \frac{\partial v_j^{\mathrm{b}}}{\partial x_j}+\frac{n}{\beta}\frac{\partial p^{\mathrm{b}}}{\partial t}. \tag{A 16}$$

The dominant part of (A 16) is

$$k\frac{\partial^2 p^{\mathrm{b}}}{\partial x_3^2} = \frac{\partial v_3^{\mathrm{b}}}{\partial x_3}+\frac{n}{\beta}\frac{\partial p^{\mathrm{b}}}{\partial t}. \tag{A 17}$$

Integrating (A 15*b*) in x_3, then substituting into (A 17), one gets Terzaghi's equation

$$k\frac{\partial^2 p^{\mathrm{b}}}{\partial x_3^2} = \left(\frac{n}{\beta}+\frac{1}{G}\frac{1-2\nu}{2(1-\nu)}\right)\frac{\partial p^{\mathrm{b}}}{\partial t}. \tag{A 18}$$

Clearly from the above equation, the scale of the boundary-layer thickness is

$$\delta = (kt)^{\frac{1}{2}}\left(\frac{n}{\beta}+\frac{1}{G}\frac{1-2\nu}{2(1-\nu)}\right)^{-\frac{1}{2}}. \tag{A 19}$$

Furthermore, since $O(p^{\mathrm{b}}) = P$, (A 15*b*) implies

$$v_3^{\mathrm{b}} = O\left(\frac{P\delta}{Gt}\right), \tag{A 20}$$

which is small as compared to outer vertical velocities (cf. (A 6)). The horizontal velocities v_1^{b} and v_2^{b} are even smaller, from (A 14). From Darcy's law (A 2), it is seen that using these boundary-layer scales, all terms in (A 2) are of the same orders of magnitude in all directions. In particular, water-velocity components are of the same order as the solid-velocity components,

$$(u_i^{\mathrm{b}}, v_i^{\mathrm{b}}) \sim O\left(\frac{Pa}{Gt}\frac{\delta^2}{a^2}\right) \quad (i = 1, 2), \tag{A 21a}$$

$$(u_3^{\mathrm{b}}, v_3^{\mathrm{b}}) \sim O\left(\frac{Pa}{Gt}\frac{\delta}{a}\right). \tag{A 21b}$$

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